JOURNAL OF COMBINATORIAL THEORY, Series B 23, 174–183 (1977)

Edge Partitions of Complete Multipartite Graphs into Equal Length Circuits

E. J. COCKAYNE

Department of Mathematics, University of Victoria, Victoria, British Columbia, Canada

AND

B. L. HARTNELL

Department of Computer Science, University of Manitoba, Winnipeg, Manitoba, Canada

Communicated by the Editors

Received June 24, 1975

Values of t, x, and m are obtained for which the complete multipartite graph defined by t independent sets each of cardinality x has an edge partition into circuits of length m.

1. INTRODUCTION

Let ${}_tK_x$ $(t, x \geqslant 2)$ denote the complete undirected multipartite graph whose vertex set may be partitioned into $X_1, X_2, \ldots, X_t$ where $|X_i| = x$ for each $i = 1, \ldots, t$ and vertices a, b are adjacent if and only if they belong to distinct sets X_i, X_j of the partition.

Our problem is to discover values of t, x, and m so that the edges of ${}_tK_x$ may be partitioned into m-circuits. We denote such a decomposition by the notation ${}_tK_x \to C_m$. Three trivial necessary conditions for decomposition are the following:

(1) $m \leqslant tx$,

(2) each vertex of ${}_tK_x$ has even degree, i.e., $x(t-1)$ is even,

and

(3) the number of edges of ${}_tK_x$ (namely, $\binom{t}{2} x^2$} is divisible by m.

In Section 2 we present lemmas useful in obtaining decompositions based on a special representation of ${}_tK_x$ and the idea of edge length. This approach has been used by Kotzig [5] and Rosa [6, 7], and later by Hartnell [4], in

ISSN 0095-8956

decomposing complete graphs and digraphs into circuits. The following section contains extension results of the form: A decompsition and certain extra conditions imply further decompositions. The main results are contained in Section 4. We should emphasize that our study is of an introductory nature and there is much scope for further research. In particular, many of our decompositions are obtained by application of the theory of block designs and it is felt that researchers with more experience in this area may well be able to extend our work fairly quickly!

2. General Lemmas

In this and later sections we use the following labeling scheme for ${}_tK_x$. Let the vertices of ${}_tK_x$ be $v_1, v_2, \ldots, v_{tx}$ and define *the length of* v_i, v_j as $\min\{|i-j|, tx-|i-j|\}$. Let v_i, v_j be adjacent if and only if the length of v_i, v_j is not divisible by t.

The lengths of the edges of ${}_tK_x$ are all integers in the set $\{1, \ldots, [tx/2]\}$ which are not divisible by t. If z is such an edge length and $z \neq tx/2$, there are precisely tx edges in ${}_tK_x$ having length z. If $z = tx/2$, there are $tx/2$ edges of length z. The t disjoint independent sets of ${}_tK_x$ with this labeling are

$$X_i = \{v_i, v_{i+t}, v_{i+2t}, \ldots, v_{i+(x-1)t}\}, \qquad i = 1, \ldots, t.$$

This representation shows that ${}_tK_x$ is a special case of a class of graphs called star polygons [2].

By the *turning of an edge* $[v_i, v_j]$ of ${}_tK_x$ we mean the increasing of both indices by 1, whereby we obtain the edge $[v_{i+1}, v_{j+1}]$ of ${}_tK_x$ from $[v_i, v_j]$ (the indices are taken modulo tx). By the *turning of a circuit* we mean the simultaneous turning of all edges of the circuit.

The ideas of length and turning were basic in the construction of edge partitions of complete directed graphs into directed circuits in [4].

Lemma 1. *Let $\{n_1, \ldots, n_d\}$ be a set of distinct lengths of edges of ${}_tK_x$ not containing the length $tx/2$, such that $\sum_{i=1}^{d} n_i = s$ where $tx = rs$ for some r. Then there exists a dr-circuit C in ${}_tK_x$ which may be turned through $s-1$ positions yielding, with C itself, s edge-disjoint dr-circuits in ${}_tK_x$ which include all edges of ${}_tK_x$ whose lengths are in $\{n_1, \ldots, n_d\}$.*

Proof. Consider C, the dr-circuit with the vertex sequence given below. The subscripts are reduced modulo tx to the set of residues $\{1, \ldots, tx\}$.

$$v_{tx}, v_{n_1}, v_{n_1+n_2}, \ldots, v_{n_1+n_2+\cdots+n_d=s},$$

$$v_{s+n_1}, v_{s+n_1+n_2}, \ldots, v_{2s}, \ldots,$$

$$v_{(r-1)s+n_1}, v_{(r-1)s+n_1+n_2}, \ldots, v_{rs=tx}.$$

We first show that in turning C $s-1$ times, no edge duplications occur. Since edge length is preserved under the turning operation, if the same edge of length n_j appears in C turned through m_1 positions and in C turned through m_2 positions where $0 \leqslant m_1 < m_2 \leqslant s-1$, then for some $\{k_1, k_2\} \subseteq \{0,..., r-1\}$ we have the unordered pair equality

$$\{v_{k_1 s+\Sigma_{i=1}^{j-1} n_i+m_1}, v_{k_1 s+\Sigma_{i=1}^{j} n_i+m_1}\} = \{v_{k_2 s+\Sigma_{i=1}^{j-1} n_i+m_2}, v_{k_2 s+\Sigma_{i=1}^{j} n_i+m_2}\}. \tag{1}$$

If these are equal as ordered pairs then

$$k_1 s + \sum_{i=1}^{j-1} n_i + m_1 \equiv k_2 s + \sum_{i=1}^{j-1} n_i + m_2 \pmod{rs}.$$

Therefore, $(k_1 - k_2)s + (m_1 - m_2) = qs$ for some integer q. We deduce s divides $m_1 - m_2$, a contradiction.

Hence to satisfy (1)

$$k_1 s + \sum_{i=1}^{j-1} n_i + m_1 \equiv k_2 s + \sum_{i=1}^{j} n_i + m_2 \pmod{tx},$$

and

$$k_2 s + \sum_{i=1}^{j-1} n_i + m_2 \equiv k_1 s + \sum_{i=1}^{j} n_i + m_1 \pmod{tx}.$$

Therefore,

$$(k_1 - k_2)s + (m_1 - m_2) \equiv n_j \pmod{tx},$$
$$(k_2 - k_1)s + (m_2 - m_1) \equiv n_j \pmod{tx}.$$

Hence, by addition, $2n_j \equiv 0 \pmod{tx}$ which is impossible since $n_j \neq tx/2$. Therefore, no edge duplications occur in the turning.

C contains r edges of length n_j for each j; hence, the s circuits produced by the turning contain rs edges of length n_j. This completes the proof of Lemma 1.

The following lemma is proved similarly and we omit the details.

LEMMA 2. *Let ${}_tK_x$ contain an r-circuit C whose edge lengths are distinct and not equal to $tx/2$. Then C may be turned through $tx - 1$ positions yielding, with C itself, tx edge-disjoint r-circuits of ${}_tK_x$ containing all edges of ${}_tK_x$ having lengths equal to the length of any edge in C.*

3. EXTENSION LEMMAS

LEMMA 3. *If ${}_tK_x \to C_m$ and there exists a balanced incomplete block design (BIBD) on v elements with $\lambda = 1$, $k = t$ (see, e.g., [3, p. 100]), then ${}_vK_x \to C_m$.*

Proof. Suppose that the disjoint independent sets of ${}_vK_x$ are labeled $1,..., v$. Then the subgraph of ${}_vK_x$ induced by the vertices of those independent sets whose labels are in a single block of the BIBD is isomorphic to ${}_tK_x$. Since $\lambda = 1$, the blocks of the BIBD yield a partition of the edges of ${}_vK_x$ into copies of ${}_tK_x$. By hypothesis, each copy of ${}_tK_x$ may be decomposed into m-circuits; hence, ${}_vK_x \to C_m$.

COROLLARY 1. *If* ${}_2K_x \to C_m$, *then* ${}_vK_x \to C_m$ *for all* $v \geqslant 3$.

Proof. The set of all 2-subsets of $1,..., v$ is a BIBD with $\lambda = 1$, $k = 2$. The result follows by applying the lemma with $t = 2$.

COROLLARY 2. *If* ${}_3K_x \to C_m$, *then for all* $v \geqslant 1$, ${}_{6v+1}K_x \to C_m$ *and* ${}_{6v+3}K_x \to C_m$.

Proof. The result follows by application of the lemma with $t = 3$ and the existence of Steiner triple systems on sets of $6v + 1$ and $6v + 3$ elements.

If ${}_tK_x$ has an edge partition into complete graphs on y vertices, we write ${}_tK_x \to K_y$.

LEMMA 4. *If* ${}_tK_x \to C_m$ *and* ${}_tK_q \to K_t$, *then* ${}_tK_{qx} \to C_m$.

Proof. Divide each of the independent sets of qx vertices in ${}_tK_{qx}$ into q blocks of x vertices. Construct a new graph with a vertex set, the set of constructed blocks and vertices adjacent if and only if the blocks came from distinct independent sets of ${}_tK_{qx}$. The constructed graph is isomorphic to ${}_tK_q$. If we convert back from the constructed ${}_tK_q$ to the original ${}_tK_{qx}$, then each K_t in an edge partition of ${}_tK_q$ into K_t's corresponds to a ${}_tK_x$ in ${}_tK_{qx}$. Hence, ${}_tK_{qx}$ has an edge partition into ${}_tK_x$'s. But ${}_tK_x \to C_m$; therefore, ${}_tK_{qx} \to C_m$.

COROLLARY 1. *If* ${}_2K_x \to C_m$, *then* ${}_2K_{qx} \to C_m$ *for all* q.

Proof. Immediate from the lemma with $t = 2$ and the fact that ${}_2K_q \to K_2$.

COROLLARY 2. *If* ${}_3K_x \to C_m$, *then* ${}_3K_{qx} \to C_m$ *for all* $q \geqslant 1$.

Proof. In [8, p. 610], Rosa and Kotzig show that ${}_3K_q \to K_3$ for $q \neq 2, 6$. However, ${}_3K_2 \to K_3$ and ${}_3K_6 \to K_3$ (see Theorem 5) as well. The result follows by use of the Lemma with $t = 3$.

4. The Principal Results

4.1. *The Case* $m = 4$

In this section we show that ${}_tK_x \to C_4$ if and only if the necessary conditions given in the introduction are satisfied. Formally, we prove

THEOREM 1. *${}_tK_x \to C_4$ if and only if $x(t-1)$ is even and 4 divides $\binom{t}{2} x^2$.*

Proof. Assume x is even. Trivially, ${}_2K_2 \to C_4$ and hence by Corollary 1 to Lemma 4, ${}_2K_{2q} \to C_4$. Now Corollary 1 to Lemma 3 asserts that ${}_tK_{2q} \to C_4$ for all $t \geqslant 3$ as required.

If $x = 2a + 1$, the necessary conditions insist that $t = 2k + 1$ for some k and that $\frac{1}{2}(2k+1)\, 2k(2a+1)^2 \equiv 0 \pmod 4$. We deduce $k \equiv 0 \pmod 4$ and hence $t = 8p + 1$ for some p.

Using the representation of Section 2, the set of edge lengths of ${}_{8p+1}K_{2a+1}$ is

$$\{1, 2,..., 8p, (8p+1)+1, (8p+1)+2,..., (8p+1)+8p,..., \\ a(8p+1)+1, a(8p+1)+2,..., a(8p+1)+4p\}.$$

Partition this set into the following sets each containing four consecutive integers:

$$\{1, 2, 3, 4\}, \{5, 6, 7, 8\},..., \{a(8p+1)+4p-3, \\ a(8p+1)+4p-2, a(8p+1)+4p-1, a(8p+1)+4p\}.$$

Consider any such set say $\{y+1, y+2, y+3, y+4\}$. Using Lemma 2, by turning the 4-circuits $v_1, v_{1+y+4}, v_2, v_{2+y+1}, v_1$ through $tx - 1$ positions, we form tx edge-disjoint 4-circuits and exhaust all edges having these lengths. Repetition of this procedure for all y yields the desired result, ${}_{8p+1}K_{2a+1} \to C_4$, and the theorem is proved.

4.2. *The Case* $m = 2^r$ *and* x *Odd*

THEOREM 2. *${}_tK_x \to C_{2^r}$ (x odd and $r \geqslant 2$) if and only if the necessary conditions hold.*

Proof. This proof is a generalization of the second part of the proof of Theorem 1. Since $x(t-1)$ is even, t is odd. If $x = 2a+1$, the other necessary condition implies that $\binom{t}{2}(2a+1)^2$ is divisible by 2^r. We deduce $(t-1)/2 \equiv 0 \pmod{2^r}$, i.e., for some t_1, $t = 2^{r+1} \cdot t_1 + 1$. The set of edge lengths of ${}_{2^{r+1}\cdot t_1+1}K_{2a+1}$ is

$$\{1, 2,..., 2^{r+1}\cdot t_1, \\ 1(2^{r+1}\cdot t_1+1)+1, 1(2^{r+1}\cdot t_1+1)+2,..., 1(2^{r+1}\cdot t_1+1)+2^{r+1}\cdot t_1, \\ \vdots \\ a(2^{r+1}\cdot t_1+1)+1, a(2^{r+1}\cdot t_1+1)+2,..., a(2^{r+1}\cdot t_1+1)+2^r\cdot t_1\}.$$

We partition this set into sets of 2^r consecutive integers. Consider some such set, say, $\{a_1, a_2, \dots, a_{2^r}\}$, where $a_1 < a_2 < \cdots < a_{2^r}$. Form the following 2^r-circuit:

$$v_1, v_{1+a_1}, v_{1+a_1-a_2}, v_{1+a_1-a_2+a_3}, \dots,$$

$$v_{1+a_1-a_2+\cdots-a_{2^{r-1}}}, v_{1+a_1-a_2+\cdots-a_{2^{r-1}}+a_{2^{r-1}+2}},$$

$$v_{1+a_1-a_2+\cdots+a_{2^{r-1}+2}-a_{2^{r-1}+3}}, \dots, v_{1+a_1-a_2+\cdots+a_{2^r}},$$

$$v_1 .$$

By turning this circuit, we exhaust the edge lengths belonging to the set $\{a_1, a_2, \dots, a_{2^r}\}$ (Lemma 2). Repetition of this procedure for each set yields the desired decomposition.

4.3. *Prime Power Factors*

This section contains decompositions of ${}_tK_x$ into m-circuits when x and m contain certain prime power factors.

THEOREM 3. *For p a prime, ${}_pK_{p^r} \to C_{p^{r+1}}$ [for p even, $r \geqslant 1$; for p odd, $r \geqslant 0$].*

Proof. The set of edge lengths of ${}_pK_{p^r}$ is

$$S = \{1, 2, 3, \dots, p-1, p+1, p+2, \dots, p+(p-1), 2p+1, \dots, [(p^{r+1}-1)/2]\}.$$

Let $l \in S$. Since l is relatively prime to p^{r+1}, we can form the following p^{r+1}-circuit:

$$v_{p^{r+1}}, v_{p^{r+1}+l}, v_{p^{r+1}+2l}, v_{p^{r+1}+3l}, \dots, v_{p^{r+1}-l} .$$

This circuit exhausts all edges of length l in ${}_pK_{p^r}$. Repeating for each $l \in S$, we decompose ${}_pK_{p^r}$ into p^{r+1}-circuits.

COROLLARY 1. ${}_tK_{2^r\cdot q} \to C_{2^{r+1}}$.

Proof. ${}_2K_{2^r} \to C_{2^{r+1}}$ by Theorem 3. By Corollary 1 of Lemma 4 we obtain ${}_2K_{2^r\cdot q} \to C_{2^{r+1}}$. Hence by Corollary 1 to Lemma 3, we have ${}_tK_{2^r\cdot q} \to C_{2^{r+1}}$ as required.

COROLLARY 2. ${}_{6t+1}K_{3^r\cdot q} \to C_{3^{r+1}}$ *and* ${}_{6t+3}K_{3^r\cdot q} \to C_{3^{r+1}}$ *(for $q \geqslant 1$ and $r \geqslant 0$).*

Proof. By the theorem, ${}_3K_{3^r} \to C_{3^{r+1}}$. Hence by Corollary 2 to Lemma 4, ${}_3K_{3^r\cdot q} \to C_{3^{r+1}}$ ($q \geqslant 1$). The result now follows from Corollary 2 to Lemma 3.

THEOREM 4. *For $s \geqslant 0$ and p prime, ${}_pK_{p^{r+s}} \to C_{p^{r+1}}$.*

Proof. There exists a resolvable BIBD with $v = p^{2\alpha}$, $k = p^{\alpha}$ and $\lambda = 1$ (see [3, p. 198]). Let $B_1, \ldots, B_k$ be a set of blocks of this design which form a complete replication of the v elements and let Y be the set of remaining blocks. Since $\lambda = 1$, the graph with vertex set $\{1, \ldots, v\}$ in which a is adjacent to b if and only if $\{a, b\}$ is contained in an element of Y, is isomorphic to ${}_kK_k$ and the k independent sets are precisely $B_1, \ldots, B_k$. The subgraph induced in ${}_kK_k$ by the vertices of any block of Y, is a copy of K_k and the condition $\lambda = 1$ ensures that each edge of ${}_kK_k$ is in precisely one of these induced subgraphs. We conclude that ${}_kK_k \to K_k$.

By Theorem 3, ${}_pK_{p^r} \to C_{p^{r+1}}$ and the above yields ${}_pK_p \to K_p$. Therefore, using Lemma 4 we deduce ${}_pK_{p^{r+1}} \to C_{p^{r+1}}$. Repetition of this argument yields ${}_pK_{p^{r+s}} \to C_{p^{r+1}}$ for all $s \geqslant 0$.

4.4. *Further Application of Block Designs*

THEOREM 5. *If $\exists(k-2)$ mutually orthogonal latin squares of order n, then*

$$ {}_{k'}K_n \to K_{k'} \qquad \text{for} \quad k' \leqslant k. $$

Proof. The existence of $k-2$ mutually orthogonal latin squares of order n implies the existence of an orthogonal array OA(n, k). But an OA(n, k) is essentially the same as a system $T_0(k, n)$ (Hanani's transversal system) whose existence immediately yields

$$ {}_kK_n \to K_k \qquad \text{(see [3, p. 224])}. $$

Since one can obtain an OA$(n, k-1)$ from an OA(n, k), the theorem follows.

COROLLARY 1. *If ${}_4K_x \to C_m$, then ${}_4K_{xn} \to C_m$ $(n \neq 2, 6)$.*

Proof. By the theorem, ${}_4K_n \to K_4$ $(n \neq 2, 6)$ since there exist 2 mutually orthogonal latin squares of order n for $n \neq 2, 6$. But by Lemma 4, ${}_4K_x \to C_m$ and ${}_4K_n \to K_4$ imply ${}_4K_{xn} \to C_m$ as required.

The above theorem giving decompositions of ${}_tK_x$ into complete graphs may be combined with results of Rosa and Kotzig [5–7] on decomposition of complete graphs into circuits to obtain more cases in which ${}_tK_x \to C_m$. Examples are given in the next two corollaries.

COROLLARY 2. *If x has prime factorization $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$, then for all m, q satisfying $8mq \leqslant p_i^{\alpha_i}$ for each i, ${}_{8mq+1}K_x \to C_{4m}$.*

Proof. There exist $\min_{i=1,\ldots,r} p_i^{\alpha_i} - 1$ mutually orthogonal latin squares of order x. Hence by the theorem, ${}_{8mq+1}K_x \to K_{8mq+1}$, provided that $8mq \leqslant p_i^{\alpha_i}$ (for all i). But $K_{8mq+1} \to C_{4m}$, by Kotzig's result [5]. Hence ${}_{8mq+1}K_x \to C_{4m}$ as required.

In a similar manner, any result of the type $K_t \to C_m$ may be combined with the prime power application of Theorem 5 to give a result of the type ${}_tK_x \to C_m$ provided that each prime power in x is greater than or equal to $t-1$.

COROLLARY 3. *Given m, q, for x sufficiently large, ${}_{8mq+1}K_x \to C_{4m}$.*

Proof. If $N(x)$ denotes the maximum number of mutually orthogonal latin squares of order x, then $\lim_{x\to\infty} N(x) = \infty$ by [9]. Therefore, if x is chosen so that $N(x) \geqslant 8mq - 1$, then by the theorem, ${}_{8mq+1}K_x \to K_{8mq+1}$. But $K_{8mq+1} \to C_{4m}$ by [5]; hence, ${}_{8mq+1}K_x \to C_{4m}$ as required.

Similarly any result of the type $K_t \to C_m$ may be combined with the limit theorem used in Corollary 3 to show that for x sufficiently large ${}_tK_x \to C_m$.

Several further results may be obtained from known group divisible incomplete block designs (see [1]) because of the following simple equivalence (proof omitted).

LEMMA 5. *${}_tK_x \to K_s$ if and only if there exists a group divisible incomplete block design with the following parameters*

$$\begin{array}{ll} v = tx & k = s \\ b = \binom{t}{2} x^2 \Big/ \binom{s}{2} & r = (x(t-1))/(s-1) \\ n = x & \lambda_1 = 0 \\ m = t & \lambda_2 = 1 \end{array}$$

THEOREM 6. ${}_{3t+1}K_2 \to C_3$ *and* ${}_{3t}K_2 \to C_3$.

Proof. There exist Steiner triple systems with parameters $v^* = 6t + 1$ or $6t + 3$, $k^* = 3$, and $\lambda^* = 1$. Therefore, using [1, Theorem 1] we deduce the existence of a group divisible design with partial parameter list

$$v = 6t + 2 \text{ or } 6t \qquad k = 3 \quad \lambda_1 = 0 \quad \lambda_2 = 1 \quad n = 2.$$

These designs and Lemma 5 yield the desired decompositions.

THEOREM 7. *For all $p \geqslant 3$, if $2yp + 1$ is a prime or a prime power, then ${}_{2yp+2}K_{2yp} \to C_p$.*

Proof. For all prime or prime power s there exist group divisible designs with parameters

$$v = b = s^2 - 1, \quad r = k = s, \quad m = s + 1, \quad n = s - 1, \quad \lambda_1 = 0, \ \lambda_2 = 1$$

(see [1, p. 173]). Setting $s = 2yp + 1$ and applying Lemma 5 we obtain

${}_{2yp+2}K_{2yp} \to K_{2yp+1}$. But K_{2yp+1} is decomposable into p-circuits (the case $p > 3$ by [5–7], the case $p = 3$ by the existence of Steiner triple systems). Hence ${}_{2yp+2}K_{2yp} \to C_p$ as required.

4.5. *A Result on Decomposition into* 4*x*-*Circuits*

THEOREM 8. *For all* q, $x \geqslant 1$ *and* $t \geqslant 2$, ${}_tK_{2xq} \to C_{4x}$.

Proof. The set of edge lengths of ${}_2K_{4x}$ is $\{1, 3, 5, 7, \ldots, 4x - 3, 4x - 1\}$. Partition this set into the x pairs $\{1, 3\}$, $\{5, 7\}, \ldots, \{4x - 3, 4x - 1\}$. Let $\{y + 1, y + 3\}$ ($y \equiv 0 \pmod 4$) be one of the pairs of this partition. Then the following is the vertex sequence of an $8x$-circuit with edge lengths alternately $y + 3$ and $y + 1$ (we recall index addition is performed modulo $8x$).

$$\begin{array}{rl}
v_{8x}, & v_{8x+y+3} = v_{y+3} \\
v_{8x+2} = v_2, & v_{y+5} \\
v_4, & v_{y+7} \\
v_6, & v_{y+9} \\
\cdots & \\
v_{8x-2}, & v_{y+8x+1} = v_{y+1} \\
v_{8x}. &
\end{array}$$

If this circuit is turned once, two edge-disjoint $8x$-circuits are formed whose edges exhaust all edges of ${}_2K_{4x}$ of length $y + 1$ or $y + 3$. The proof is omitted. Repeating this procedure for each y, we deduce

$$ {}_2K_{4x} \to C_{8x} \tag{2}$$

The set of edge lengths of ${}_2K_{4x+2}$ is $\{1, 3, 5, \ldots, 4x + 1\}$. Partition this set into $\{1\}$, $\{3, 5\}$, $\{7, 9\}, \ldots, \{4x - 1, 4x + 1\}$. The pairs are treated as above, i.e., for each pair of lengths $y + 1$, $y + 3$, we form the following $(8x + 4)$-circuit with lengths alternately $y + 3$ and $y + 1$:

$$\begin{array}{rl}
v_{8x+4}, & v_{y+3} \\
v_2, & v_{y+5} \\
v_4, & v_{y+7} \\
\vdots & \\
v_{8x+2}, & v_{y+1} \\
v_{8x+4}. &
\end{array}$$

If for each y, we turn this circuit once, we form $2x$ $(8x + 4)$-circuits whose edges exhaust all edges of ${}_2K_{4x+2}$ except those with length 1. These remaining edges clearly form a $(8x + 4)$-circuit and we obtain

$$ {}_2K_{4x+2} \to C_{8x+4} \tag{3}$$

From (2) and (3) we deduce ${}_2K_{2x} \to C_{4x}$ and by Corollary 1 to Lemma 4 we have the desired result ${}_2K_{2xq} \to C_{4x}$.

COROLLARY 1. *${}_2K_x$ may be edge partitioned into Hamiltonian circuits if and only if the trivial necessary condition is satisfied.*

Proof. The theorem implies ${}_2K_{2x} \to C_{4x}$.

ACKNOWLEDGMENT

The authors gratefully acknowledge the support of the Canadian National Research Council.

REFERENCES

1. R. C. BOSE, S. S. SHRIKHANDE, AND K. N. BHATTACHARYA, On the construction of group divisible incomplete block designs, *Ann. Math. Stat.* **24** (1953), 167–195.
2. F. BOESCH AND A. P. SELZER, A general class of invulnerable graphs, *Networks* **2** (1972), 261–283.
3. M. HALL, "Combinatorial Theory," Ginn Blaisdell, Waltham, Mass., 1967.
4. B. HARTNELL, Decomposition of K_{xy}^*, x and y odd, into $2x$-circuits, *in* "Proceedings of the 4th Man. Conf. on Numerical Math.," October 1974, pp. 265–271.
5. A. KOTZIG, On the decomposition of a complete graph into $4K$-gons, *Mat.-Fyz. Cas.* **15** (1965), 229–233.
6. A. ROSA, On the cyclic decompositions of the complete graph into polygons with odd number of edges, *Cas. Pro. Pest. Mat.* **91** (1966), 53–63.
7. A. ROSA, On cyclic decompositions of the complete graph into $(4m + 2)$-gons, *Mat.-Fyz. Cas. Sav.* **16**, No. 4 (1966), 349–352.
8. A. ROSA AND A. KOTZIG, "Nearly Kirkman Systems," Congressus Numerantium X, pp. 607–614, Utilitas Math., Winnipeg, Manitoba, 1974.
9. S. CHOWLA, P. ERDÖS, AND E. G. STRAUS, On the maximal number of pairwise orthogonal latin squares of a given order, *Canad. J. Math.* **12** (1960), 204–208.